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Zero Divisor Graphs of Idealizations for Direct Products of Commutative Rings

Mehrdad Azadi *, Azemat Ramzanpoor Mardekheh

Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran

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Abstract

Let R_1and R_2be commutative ring, M be a prime module over R_1and R_2. Let Z(R) be the set of zero divisors of a commutative ring R. The zero divisor graph of a commutative ring R, is the simple graph with vertices Z(R)-{0}, denoted Γ(R). Two distinct vertices x and y of Z(R)-{0} are adjacent if and only if xy=0. In this paper we study the diameter of ***Γ***((R_1×R_2) (+) M) with respect to the diameters of the zero divisor graphs of R_1(+) M and R_2(+) M.

1. Introduction

For the sake of completeness, we explain some definitions and points used throughout of the paper. All rings in this paper are commutative. Let Z(R) be the set of zero divisors of R. The idea of associate graph to a commutative ring, where all elements of the ring are vertices of that graph, first appeared in [5] which deals with graph coloring . In [2], Anderson and Livingston take the nonzero zero divisors for the vertices of the graph and two vertices x,y∈ Z(R)-{0} are adjacent if and only if xy=0. The resulting graph Γ(R) is the zero divisor graph of the commutative ring R. Also the authors proved that if R is a commutative ring, then Γ(R) is connected and has diameter less than or equal to three [2, Theorem 2.3.]. In [4], M. Axtell and J. Stickles study the diameter and girth of the zero divisor graph of idealization of a commutative ring. In [10], J. Warfel describes the diameter of a zero divisor graph for a direct product R_1×R_2 with respect to the diameters of the zero divisor graphs of R_1and R_2. In this paper we study the diameter of ((R_1×R_2) (+) M) with respect to the diameters of the zero divisor graphs of R_1(+) M and R_2(+) M. Let ***Γ*** be a graph. For two arbitrary vertices x and y of graph ***Γ*** a path of length t between x and y is an ordered list of distinct vertices x=x_0,x_1,…,x_t=y such that x_(i-1) x_i are edges for all i= 1,…,t. We denote a path between x and y by x=x_0-x_1-…-x_t=y. For two vertices x and y of ***Γ***, the length of a shortest path from x to y is denoted by d(x,y). Note that if there is no path of finite length between x and y , then d(x,y)=∞. The diameter of Γ is defined as diam(***Γ***)=sup{d(x,y): x and y are vertices of ***Γ***}. The graph ***Γ*** is complete if it is connected with diameter one. We use the notation A^*to refer to the non-zero elements of A. Let M be a module over commutative ring R. Let Z_R(M) be the set of zero divisors of M over R, defined as Z_R(M)={a∈R: am=0 for some m∈ M^*}. Define (0:_R M)={r∈R:rm=0 , for all m∈M}. R-module M is called prime if whenever rm=0 either m=0 or r∈(0:_R M) [6]. The idealization of M in R, denoted by R (+)M, is a ring with the following operation:

(a,m) + (b,n)=(a+b, m+n)

(a,m) (b,n)=(ab, an+bm).

We will assume that neither the ring nor the module is trivial [8].

2. Diameter of ((R_1×R_2) (+) M)

In this section we investigate the relation between diameter of ((R_1×R_2) (+) M) with the diameters of the zero divisor graphs of R_1(+) M and R_2(+) M. In this section we let (R_1), (R_2)≠ϕ

Lemma 2.1.

Let R be a commutative ring, M be a prime module over R. Then (a,m)∈ $Z(R(+)M)^*$if and only if a∈$(0:_R M) \cup Z(R)$.

Proof. Suppose that (a,m))∈ $Z(R(+)M)^*$. If a=0, the proof is complete. Thus we assume that a≠ 0. There exists (b,n) of $Z(R(+)M)^*$such that (a,m) (b,n)=(0,0); hence ab=0 and an+bm=0. If b=0 then n≠0 and an=0 and hence a∈$(0:_R M)$, since M is a prime module. If b≠0, then a∈ $Z(R)$.

Conversely, assume that (a,m)∈ $R(+)M$ with a∈$(0:_R M) \cup Z(R)$. Let a∈ $Z(R)$, then ab=0, for some b≠ 0. If b∈$(0:_R M)$,then (a,m) (b,n)=(0,0). If b ∉ $(0:_R M)$, then there exists a non zero element x of M such that bx≠ 0, therefore (a,m) (0,bx)=(0,0). Let a∈$(0:_R M)$, then for all n of

* Corresponding Author:
E-mail address: meh.azadi@iauctb.ac.ir

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M we have an=0. Thus (a,m) (0,n)=(0,0). So in any cases we have (a,m)∈ $Z(R(+)M)^*$.

Lemma 2.2.

Let R be a commutative ring, M be a prime module over R. If $\Gamma(R) \neq \boldsymbol{\phi}$, then $\boldsymbol{\Gamma}$ $(R(+)M)$ is complete graph if and only if $Z(R)^2 = \{0\}$ and $Z(R) = (0:_R M)$.

Proof. Let $R(+)M$ is complete graph and r,s ∈ $Z(R)^*$. Hence (r,0) (s,0)=(0,0). So rs=0 and $Z(R)^2 = \{0\}$.

If there exists m∈ M^* and a∈ $Z(R)^*$ such that am≠0, then (a,0) (0,m)≠(0,0), which is a contradiction, since $\boldsymbol{\Gamma}(R(+)M)$ is complete graph. Let x∈ $(0:_R M)$, then (x,0) ∈ $Z(R(+)M)$.* Since $\Gamma(R) \neq \boldsymbol{\phi}$, so there exists a∈ $Z(R)^*$. Hence (x,0) (a,0)=(0,0), since $\boldsymbol{\Gamma}(R(+)M)$ is complete graph. Thus xa=0 implies that x∈ $Z(R)$. Thus $Z(R) = (0:_R M)$.

Conversely, let (x,m), (y,n) ∈ $Z(R(+)M)^*$. Thus x,y∈ $(0:_R M) \cup Z(R)$, by Lemma 2.1. Hence xy=0 and xn+ym=0, since $Z(R)^2 = \{0\}$ and $Z(R) = (0:_R M)$. Therefore $(R(+)M)$ is complete graph.

We need the following Lemma from [3, Theorem 3.9].

Lemma 2.3.

Let R be a commutative ring, M be a module over R. Then diam($\boldsymbol{\Gamma}(R(+)M)$)≤ 2 if and only if for all x,y∈ $Z(R) \cup Z_R$(M), either (1), there is a 0≠ $z \in (0:_R M)$ such that xz=yz=0, or (2), there is a 0≠m∈ M such that xm=ym=0.

Theorem 2.4.

Let R_1 and R_2 be commutative rings, M be a prime module over R_1 and R_2. Let diam($\boldsymbol{\Gamma}(R_1(+)M)$)= diam($\boldsymbol{\Gamma}(R_2(+)M)$)=1. Then diam ((($R_1$×$R_2$) (+) M))=1.

Proof. Suppose that $((x_1, y_1), m_1)$ and $((x_2, y_2), m_2)$ are belong to $Z((R_1 \times R_2)(+)\, M)^*$. Then by Lemma 2.1, (x_1, y_1) and (x_2, y_2) are belong to Z$(R_1 \times R_2) \cup$ (0: $_{R_1 \times R_2}$M). Since, diam($\boldsymbol{\Gamma}(R_1(+)M)$)=1, thus by Lemma 2.2, $Z(R_1)^2 = \{0\}$ and Z(R_1)= (0: $_{R_1}$M). Since, diam($\boldsymbol{\Gamma}(R_2(+)M)$)=1, thus by Lemma 2.2, $Z(R_2)^2 = \{0\}$ and Z(R_2)= (0: $_{R_2}$M). So $Z(R_1 \times R_2)^2 = \{0\}$ and Z($R_1 \times R_2$)= (0: $_{R_1 \times R_2}$M). Hence, $((x_1, y_1), m_1)\,((x_2, y_2), m_2) =$ ((0,0),0). Thus, diam (((R_1×R_2) (+) M))=1.

Lemma 2.5.

Let R_1 and R_2 be commutative rings, M be a prime module over R_1 and R_2. If diam($\boldsymbol{\Gamma}(R_i(+)M)$)> 1, for i=1,2. Then diam (((R_1×R_2) (+) M))> 1.

Proof. Without loss of generality let diam($\boldsymbol{\Gamma}(R_1(+)M)$)> 1. So by Lemma 2.2, $Z(R_1)^2 \neq 0$ or Z(R_1)≠ (0: $_{R_1}$M). If $Z(R_1)^2 \neq 0$, so there exist x_1, x_2 of $Z(R_1)^*$ such that $x_1 x_2 \neq 0$. Let y∈ $Z(R_2)^*$, so we have ((x_1, y),0), ((x_2, y),0)∈ $Z((R_1 \times R_2)(+)\, M)^*$ and ((x_1, y),0) ((x_2, y),0)=(($x_1 x_2, y^2$), 0) ≠ ((0,0), 0), so diam ($\boldsymbol{\Gamma}$((R_1×R_2) (+) M))> 1. Now let Z(R_1)≠ (0: $_{R_1}$M), so there exist $x_1 \in Z(R_1)^*$ and m∈ M^*such that $x_1 m \neq 0$, thus ((x_1, 0),0), ((0,0),m)∈ $Z((R_1 \times R_2)(+)\, M)^*$ and ((x_1, 0),0) ((0,0),m))≠ ((0,0), 0); hence diam ($\boldsymbol{\Gamma}$((R_1×R_2) (+) M))> 1.

Theorem 2.6.

Let R_1 and R_2 be commutative rings, M be a prime module over R_1 and R_2. Let diam($\boldsymbol{\Gamma}(R_1(+)M)$)= 1 and diam($\boldsymbol{\Gamma}(R_2(+)M)$)≥ 2. Then diam ((($R_1$×$R_2$) (+) M))=2.

Proof. Since diam($\boldsymbol{\Gamma}(R_2(+)M)$)≥ 2, so by Lemma 2.5 it must be that diam ($\boldsymbol{\Gamma}$((R_1×R_2) (+) M))> 1. Suppose that $((x_1, y_1), m_1)$ and $((x_2, y_2), m_2)$ are belong to $Z((R_1 \times R_2)(+)\, M)^*$. Consider x∈ $Z(R_1)^*$. Since diam($\boldsymbol{\Gamma}(R_1(+)M)$)= 1, so by Lemma 2.2, it must be that x$x_1 = x x_2 = 0$ and x$m_1 = x m_2 = 0$. Therefore $((x_1, y_1), m_1) - ((x, 0), 0)$- $((x_2, y_2), m_2)$ is a path of length 2 in $\boldsymbol{\Gamma}$((R_1×R_2) (+) M) and hence diam ($\boldsymbol{\Gamma}$((R_1×R_2) (+) M))=2.

Theorem 2.7.

Let R_1 and R_2 be commutative rings, M be a prime module over R_1 and R_2. Let diam($\boldsymbol{\Gamma}(R_1(+)M)$)= diam($\boldsymbol{\Gamma}(R_2(+)M)$)= 2. Then diam ((($R_1$×$R_2$) (+) M))=2.

Proof. By Lemma 2.5, diam (((R_1×R_2) (+) M))> 1.

Suppose that $((x_1, y_1), m_1)$ and $((x_2, y_2), m_2)$ are belong to $Z((R_1 \times R_2)(+)\, M)^*$.Since, diam($\boldsymbol{\Gamma}(R_1(+)M)$)= 2 and diam($\boldsymbol{\Gamma}(R_2(+)M)$)= 2. Thus by Lemma 2.3, one of the following cases hold.

Case 1: Let there is a non zero elements z_1 of (0: $_{R_1}$M) or z_2 of (0: $_{R_2}$M) such that $x_1 z_1 = x_2 z_1 = 0$ or $y_1 z_2 = y_2 z_2$=0. So $(x_1, y_1)(z_1, 0)$=$(x_2, y_2)(z_1, 0)$=(0,0) or $(x_1, y_1)(0, z_2)$= $(x_2, y_2)(0, z_2)$=(0,0). Therefore by Lemma 2.3, diam (((R_1×R_2) (+) M))=2, since $(z_1, 0)$ and $(0, z_2)$ are belong to (0: $_{R_1 \times R_2}$M).

Case2: Let there are non zero elements m_1 and m_2 of M such that $x_1 m_1 = x_2 m_1 = 0$ and $y_1 m_2 = y_2 m_2$=0. Since M is prime, so for all m of M we have $x_1 m = x_2 m = 0$ and $y_1 m = y_2 m$=0. Hence, for all m of M, $(x_1, y_1)m$ =$(x_2, y_2)m$=(0,0) and thus by Lemma 2.3, diam ($\boldsymbol{\Gamma}$((R_1×R_2) (+) M))=2.

Theorem 2.8.

Let R_1 and R_2 be commutative rings, M be a prime module over R_1 and R_2. Let diam($\boldsymbol{\Gamma}(R_1(+)M)$)=2 and diam($\boldsymbol{\Gamma}(R_2(+)M)$)= 3. Then diam ((($R_1$×$R_2$) (+) M))= 2 or 3.

Proof. Since diam($\boldsymbol{\Gamma}(R_1(+)M)$)=2, so by Lemma 2.5, we have diam ($\boldsymbol{\Gamma}$((R_1×R_2) (+) M))> 1. Consider $((x_1, y_1), m_1)$ and $((x_2, y_2), m_2)$ are belong to $Z((R_1 \times R_2)(+)\, M)^*$. Since diam($\boldsymbol{\Gamma}(R_1(+)M)$)=2, so by Lemma 2.3, one of the following cases hold.

Case 1: There exists non zero element z_1 of (0: $_{R_1}$M) such that $x_1 z_1 = x_2 z_1 = 0$, for all $x_1, x_2 \in$ Z(R_1)∪ (0: $_{R_1}$M). Then $(x_1, y_1)(z_1, 0)$= $(x_2, y_2)(z_1, 0)$=(0,0). Therefore by Lemma 2.3, diam (((R_1×R_2) (+) M))=2.

Case 2: Let there are non zero element m_1 of M such that $x_1 m_1 = x_2 m_1 = 0$, for all $x_1, x_2 \in$ Z(R_1) ∪ (0: $_{R_1}$M). Since M is prime, so for all m of M we have $x_1 m = x_2 m = 0$. Assume that diam (((R_1×R_2) (+) M))=2. Then there must exists ((a,b),m)∈ $Z((R_1 \times R_2)(+)\, M)^*$ such that

$((x_1, y_1), m_1)$ ((a,b),m)= $((x_2, y_2), m_2)$ ((a,b),m)=((0,0),0). Thus $x_1 a = x_2 a = 0$ and $y_1 b = y_2 b = 0$; hence a=0, since Case 1 is not hold and b=0 since diam($\boldsymbol{\Gamma}(R_2(+)M)$)= 3. Also, $x_1 m_1 + y_1 m_1$=0 and $x_2 m_2 + y_2 m_2$=0 implies that $y_1 m_1 = y_2 m_2 = 0$, since $x_1 m_1 = x_2 m_1 = 0$. So for all m of M we have $y_1 m_1 = y_2 m_2 = 0$, since M is prime. Hence by Lemma 2.3, diam($\boldsymbol{\Gamma}(R_2(+)M)$)= 2, which is a contradiction. Thus diam (((R_1×R_2) (+) M))=3. Therefore in any cases diam (((R_1×R_2) (+) M))= 2 or 3.

Theorem 2.9.

Let R_1 and R_2 be commutative rings, M be a prime module over R_1 and R_2. Let diam($\boldsymbol{\Gamma}(R_1(+)M)$)= diam($\boldsymbol{\Gamma}(R_2(+)M)$)= 3. Then diam ($\Gamma$(($R_1$×$R_2$) (+) M))= 3 if and only if for m∈ M and (x_1, y_1), (x_2, y_2) ∈ Z($R_1 \times R_2$) ∪ (0: $_{R_1\times R_2}$M), then $(x_1, y_1)m$ =$(x_2, y_2)m = 0$ implies that m=0.

Proof. Assume that diam (((R_1×R_2) (+) M))=3. If there is a m of M^*such that $(x_1, y_1)m$ =$(x_2, y_2)m = 0$, for (x_1, y_1), (x_2, y_2) ∈ $Z(R_1 \times R_2)$ ∪ (0: $_{R_1\times R_2}M$), then by Lemma 2.3, diam ($\boldsymbol{\Gamma}$((R_1×R_2) (+) M))=2, which is a contradiction. Conversely, assume that diam ($\boldsymbol{\Gamma}$((R_1×R_2) (+) M))=2, then by Lemma 2.3, there is 0≠(z_1, z_2) ∈ (0: $_{R_1\times R_2}M$) such that $(x_1, y_1)(z_1, z_2) = (x_2, y_2)(z_1, z_2)$=(0,0), for all (x_1, y_1), (x_2, y_2) ∈ Z($R_1 \times R_2$)∪ (0: $_{R_1\times R_2}$M). Hence $x_1 z_1 = x_2 z_1 = 0$ and $y_1 z_2 = y_2 z_2 = 0$, then by Lemma 2.3, diam($\boldsymbol{\Gamma}(R_1(+)M)$)=diam($\boldsymbol{\Gamma}(R_2(+)M)$)= 2, which is a contradiction. So diam (((R_1×R_2) (+) M))= 3.

10. Conclusions

In this paper a review of the zero divisor graph of idealization is presented. We completely characterized the diameter of the graph $\boldsymbol{\Gamma}$((R_1×R_2) (+) M). We studied when this graph is a complete graph and has diameter 2 or 3.

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